

The "recoil" correction  $\sim m\alpha^6$  to hyperfine splitting of positronium ground state.

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### Abstract

The "recoil" correction of order  $m\alpha^6$  to the hyperfine splitting of positronium ground state was found. The formalism employed is based on the noncovariant perturbation theory in QED. Equation for two-particle component of full (many-body) wave function is used, in which effective Hamiltonian depends on the energy of a system. The effective Hamiltonian is not restricted to the nonrelativistic region, so there is no need in any regularization. To evaluate integrals over loop momenta, they are divided into "hard" and "soft" parts, coming from large and small momenta respectively. Soft contributions were found analytically, and hard ones are evaluated by numerical integration. Some soft terms due to the retardation cancel each other. To calculate the "hard" contributions, a great number of noncovariant graphs is replaced by only a few covariant ones. The hard contribution was found in two ways. The first way is to evaluate contributions of separate graphs, using the Coulomb gauge. The second one is to calculate full hard contribution as a whole using the Feynman gauge. The final result for the "recoil" correction is  $0.381(6)m\alpha^6$  and agrees with those of [1, 2]. Diagram-to-diagram comparison with the revised results of [3] was done. All the results agree, so the "recoil" correction is now firmly established. This means a considerable disagreement with the experimental data.

## 1. INTRODUCTION.

The treatment of relativistic bound states is one of challenging tasks of Quantum Electrodynamics. Positronium, the bound state of the electron and the positron, is one of the most appropriate objects for theoretical and experimental study of relativistic bound states. Because of the small masses of its constituents, effects of the strong and the weak interaction are negligible compared with the accuracy of current experiments on the positronium spectroscopy. On the other hand this experimental accuracy is good enough for comparison with results of modern theoretical investigations, which reached accuracy level  $\sim m\alpha^6$

for contributions to the energy. Therefore, now positronium spectrum can be investigated within QED framework.

Now the most accurately measured positronium property is hyperfine splitting of the ground state, i.e. energy difference between the  $1^3S_1$  and  $1^1S_0$ -states, denoted below as  $\Delta\nu$ . The two best experimental results for this value are

$$\Delta\nu = 203\,387.5(1.6)\,MHz, \quad (1)$$

$$\Delta\nu = 203\,389.10(0.74)\,MHz; \quad (2)$$

obtained in [4, 5] and [6] respectively.

Calculation of  $\Delta\nu$  has a long history; contributions  $\sim m\alpha^4$ ,  $\sim m\alpha^5$ ,  $\sim m\alpha^6 \ln \alpha$  were found in [7, 8, 9, 10, 11, 12]; their sum equals

$$m\alpha^4 \left[ \frac{7}{12} - \frac{\alpha}{\pi} \left( \frac{8}{9} + \frac{1}{2} \ln 2 \right) - \frac{5}{24} \alpha^2 \ln \alpha \right] = 203\,400.29\,MHz. \quad (3)$$

To compare experimental results (1,2) with the theoretical one, the contributions  $\sim m\alpha^6$  (without the logarithm) are to be obtained. An essential progress in their calculation was made only recently.

There are several sets of corrections  $\sim m\alpha^6$  to the hyperfine splitting, that are different in origin and may be calculated independently. First, these are three sets consisting of contributions arising from the one-, two-, and three-photon annihilation; they were found in [13, 14], [15] and [16, 17] respectively. Second, there are contributions associated with radiative corrections to the Breit potential, i.e. those of formal order  $(Z\alpha)^4 \alpha^2 m$ , found in [18, 19]. Third, there exist "recoil" corrections, i.e. those coming from the graphs in which all photon lines connect two fermion ones; these corrections are of conventional order  $(Z\alpha)^n m$  (here  $Ze$  is the charge of one of the particles;  $Z = 1$  in positronium). Finally, there are radiative-recoil (i.e.  $\sim (Z\alpha)^5 \alpha m$ ) corrections found in [20, 21, 22].

Below the sum of contributions  $\sim m(Z\alpha)^6$ ,  $\sim m(Z\alpha)^6 \ln \alpha$  to  $\Delta\nu$  is denoted as  $\Delta\nu_{rec}$ . Calculation of its nonlogarithmic part is the most difficult step of calculation of  $\Delta\nu$ . This was the subject of works [23, 1, 3, 2]. The present work is also devoted to calculation of  $\Delta\nu_{rec}$ .

The results of the three first works, namely [23, 1, 3], were all different. The result of work [2] coincides essentially with the result of [1] (they are equal to  $0.3763m\alpha^6$  and  $0.3767(17)m\alpha^6$  respectively). The result of the present work is  $0.381(6)m\alpha^6$  and also coincides, within its accuracy, with the results of [1, 2]. Recently the result of [3] was corrected by its authors. Now it is equal to  $0.3764(35)m\alpha^6$  and agrees with the results mentioned above.

In [23] the so-called NRQED (nonrelativistic QED), an effective field theory equivalent to QED, was formulated and then applied to calculation of  $\Delta\nu_{rec}$ ; nonlogarithmic part of the result equals  $0.167(33)m\alpha^6$ . However, just now the preliminary result was obtained in the NRQED framework [24], which is consistent with that of [1, 2]. In [3] the calculation was performed by Bethe-Salpeter formalism. In [1] the calculation was done by an effective

Hamiltonian approach; the method employed in [2] in fact also uses an effective Hamiltonian. Difference between [1] and [2] is in different regularizations used; besides that, the result of [1] was obtained by numerical integration, whereas that of [2] is analytic. The effective Hamiltonian approach is essentially a combination of ideas of the NRQED and the old-fashioned noncovariant perturbation theory in QED. The latter was used first for QED bound state calculations in [25].

The present paper also employs a formalism based on old-fashioned noncovariant ("time-ordered") perturbation theory for QED. Starting from the Shrödinger equation for full (many-body) wave function one easily obtains an equation containing only two-particle part of the wave function, in which effective Hamiltonian depends on the energy of a system. Then the effective Hamiltonian is divided into unperturbed part and perturbation, zeroth approximation for the two-particle wave function is found, and using them corrections to the energy levels are calculated by means of usual Rayleigh-Shrödinger perturbation theory. This calculation consist essentially in finding of expectation values of operators corresponding to graphs of the noncovariant perturbation theory over the "unperturbed" wave function. To evaluate the integrals over loop momenta each of the integrands is divided into two parts, "hard" and "soft"; here terms "hard" and "soft" mean that in the desired order a "hard" part entirely arises from the region of momenta  $\sim m$ , whereas for a "soft" part the region of momenta  $\sim \alpha m$  is also essential. This decomposition is performed in such a way that "soft" contributions may be easily found analytically, and "hard" ones are found by numerical integration. To calculate the "hard" contributions with the required accuracy external legs of the graphs may be set on the mass shell, which allows to replace the sum of a great number of noncovariant graphs by the sum of only a few covariant ones.

The method of calculation described above differs from that of [1] and [2] in the way effective Hamiltonian is defined. In [1] and [2], effective Hamiltonian is constructed so as to reproduce the scattering amplitudes, whereas in the present paper it is immediately derived from the full QED Hamiltonian. Besides that, the effective Hamiltonian used in the present work is not restricted to the nonrelativistic region, in difference with [1] and [2], so there is no need in regularization of the effective Hamiltonian or matrix elements it enters.

The plan of the rest of the paper is following. In section 2, I briefly describe the formalism used in the paper, i.e. the effective "Shrödinger equation" for two-particle component of the wave function, "zeroth" approximation to the wave function, perturbation theory formulas for the calculation of  $\Delta\nu$ ; these formulas are transformed into form of expectation values over the nonrelativistic Coulomb wave function. In section 3, I describe the general method of calculation of these expectation values, namely, the explicit manner to divide them into "soft" and "hard" parts; correspondence between these values and covariant graphs (and also scattering amplitudes on mass shell) is established. In section 4 this general method is applied to contributions of various sets of graphs in turn. Section 5 contains the description of the procedure of numerical evaluation of "hard" contributions, the checking of this procedure, and comparison of the results obtained with the results of [1, 2, 3]. Section 6 consist of summary including comparison with experimental results.

## 2. FORMALISM.

The starting point of the formalism used in the present paper is the Shrödinger equation

$$H|\psi_N\rangle = E|\psi_N\rangle, \quad (4)$$

where the positronium wave function  $|\psi_N\rangle$  includes components with different number of particles. In (4)  $H$  is the full QED Hamiltonian,  $H = H_0 + V_N$ , where  $H_0$  is the free Hamiltonian,  $V_N$  is the interaction.

Direct use of the many-body wave function is inconvenient. However in the lowest order in  $\alpha$  the only nonvanishing component of  $|\psi_N\rangle$  is two-particle one, which is denoted as  $|e\rangle$ . It is easy to obtain an equation containing only  $|e\rangle$ . Let  $P_e$  be projector onto the two-particle subspace,  $P_a = 1 - P_e$ ,

$$|e\rangle \equiv P_e|\psi_N\rangle, \quad |a\rangle \equiv P_a|\psi_N\rangle.$$

From the Shrödinger equation one gets

$$P_e H P_e |e\rangle + P_e H P_a |a\rangle = E |e\rangle, \quad (5)$$

$$P_a H P_e |e\rangle + P_a H P_a |a\rangle = E |a\rangle. \quad (6)$$

Expressing  $|a\rangle$  from (6) and substituting it into (5) results in

$$(P_e H P_e + V_a(E) - E) |e\rangle = 0, \quad (7)$$

where

$$V_a(E) = P_e H P_a \frac{1}{(E - P_a H P_a)} P_a H P_e.$$

Energy levels which could be obtained from (7) are derived in a form of perturbative expansion. For this purpose "Hamiltonian"  $P_e H P_e + V_a(E)$  is divided into the unperturbed part  $H^{(0)} \equiv P_e H_0 P_e + V_0$  and the perturbation  $V(E) \equiv V_a(E) + P_e V_N P_e - V_0$ , where  $V_0$  is the usual nonrelativistic Coulomb interaction. Contributions to the energy, which are of 1-st, 2-nd, and 3-rd order in  $V$ , have leading orders (besides logarithms)  $m\alpha^4$ ,  $m\alpha^5$  and  $m\alpha^6$  respectively.

Clearly  $V(E) + V_0 = V_a(E) + P_e V_N P_e$  is the two-particle irreducible kernel for the Green function appearing in the noncovariant perturbation theory (for transitions between two-particle states). Its expansion in  $\alpha$  is

$$V(E) + V_0 = P_e V_N \left( 1 + \frac{P_a}{E - H_0} V_N + \frac{P_a}{E - H_0} V_N \frac{P_a}{E - H_0} V_N + \dots \right) P_e$$

and corresponds to the set of two-particle irreducible graphs of the noncovariant perturbation theory (here and below "irreducible" is meant in noncovariant sense, for instance, graph shown in fig.1 is irreducible).

Since diagram technique for the noncovariant theory is not of common use, its rules are described here. External fermion lines correspond to positive-energy spinors normalized as

$u^+u = 1$  (for convenience, the fermion-fermion channel, instead of the fermion-antifermion one, is considered). Internal fermion lines are described by the projectors

$$\Lambda^\pm(\mathbf{p}) = (\varepsilon_p \pm (\gamma_0 + \alpha\mathbf{p})) / (2\varepsilon_p) \quad , \quad \varepsilon_p \equiv \sqrt{\mathbf{p}^2 + m^2} \quad ;$$

factors corresponding to external lines may be written in form

$$u(\mathbf{p}) = ((2\varepsilon_p)/(\varepsilon_p + m))^{1/2} \Lambda^+(\mathbf{p}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} w \quad ,$$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is  $(2 \times 4)$  matrix,  $w$  is a two-spinor normalized as  $w^+w = 1$ . The two fermions are assumed to have charges of opposite sign; then Coulomb photon with the momentum  $\mathbf{q}$  gives the factor  $-4\pi\alpha/\mathbf{q}^2$ , and magnetic (transverse) photon gives the factor  $(-4\pi\alpha)(\vec{\alpha}_1\vec{\alpha}_2 - (\vec{\alpha}_1\mathbf{q})(\vec{\alpha}_2\mathbf{q})/q^2)/2q$ , (in the noncovariant perturbation theory energy of virtual photon is equal to its momentum). Every negative-energy projector  $\Lambda^-$  gives extra factor  $(-1)$ . Factor  $(E - E_k + i0)^{-1}$  corresponds to each intermediate state, where  $E_k$  is energy of the intermediate state,  $E$  is full energy of a system. Loop momenta  $\mathbf{q}_i$  which remain undetermined lead to the integration  $d^3q_i / (2\pi)^3$ .

In the rest of the paper all operators and wave functions refer to the two-particle subspace. All the calculations are performed in the center-of-mass frame, so the only variable except spins is  $\mathbf{p}$ , i.e. momentum of particle 1 in the c.m.f.. Eigenstates of  $\mathbf{p}$  are normalized as usual, according to  $\langle \mathbf{p}' | \mathbf{p} \rangle = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}')$ , kernel of an arbitrary operator  $X$  is denoted as  $X(\mathbf{p}, \mathbf{p}') \equiv \langle \mathbf{p}' | X | \mathbf{p} \rangle$ .

The kernel  $V(\mathbf{p}, \mathbf{p}'; E) + V_0(\mathbf{p}, \mathbf{p}')$  is given by the sum of matrix elements corresponding to all the irreducible graphs for initial and final states having momenta  $\mathbf{p}$  and  $\mathbf{p}'$  respectively. The tree diagram with Coulomb photon corresponds to operator  $V_c$ , and that with magnetic photon corresponds to  $V_m$ . Let us write  $V$  in the form

$$V(E) = V_1 + V_2 + V_3 + \dots \quad , \quad (8)$$

where

$$V_1 = V_{1c} + V_m \quad , \quad V_{1c} \equiv V_c - V_0 \quad ,$$

$V_2$ ,  $V_3$  and so on correspond to sums of the irreducible graphs having two, three, and more, photon lines respectively. Being expanded in  $v/c$  up to the second order,  $H^{(0)} + V_1$  gives the Breit Hamiltonian.

Consider, for example, the first-order in  $V$  correction to energy levels. The first term in (8) gives to this correction contribution of formal order  $\sim m\alpha^4$ , the second term gives  $\sim m\alpha^5$ , and so on. However in fact this expansion does not converge: for instance, spin-independent correction to the energy, arising from  $V_2$ ,  $V_3$  and further terms of expansion (8), are all of order  $\sim m\alpha^5$ . This takes place due to graphs shown in fig. 2: (a), (b), (c), and so on, because these graphs produce "ultrasoft" contributions, i.e. those coming from the region where momentum of the virtual magnetic photon is  $\sim m\alpha^2$ . Fortunately for the hyperfine splitting the "rest term" of this expansion corresponds to the graphs first of which is graph shown at fig 2(c), and has order  $\sim m\alpha^7$ .

When deriving formulas for corrections to energy levels the ground state of positronium may be regarded as non-degenerate, because S-states with  $\sigma = 0$  and  $\sigma = 1$  (where  $\sigma$  is the positronium full spin) do not mix. For perturbation  $V$  depending on full energy of a system, correction to non-degenerate energy level number  $n$  reads

$$\begin{aligned} \Delta E_n = & V_{nn} + \sum_{m \neq n} \frac{|V_{nm}|^2}{E_n - E_m} + \sum_{k \neq n} \sum_{m \neq n} \frac{V_{nm} V_{mk} V_{kn}}{(E_m - E_n)(E_k - E_n)} + \\ & + V_{nn} \frac{\partial V_{nn}}{\partial E_n} + \frac{\partial}{\partial E_n} \left( V_{nn} \sum_{m \neq n} \frac{|V_{nm}|^2}{E_n - E_m} \right) + \frac{\partial}{\partial E_n} \left( \frac{1}{2} V_{nn}^2 \frac{\partial V_{nn}}{\partial E_n} \right), \end{aligned} \quad (9)$$

where summation over continuous part of the spectrum, as well as over discrete one, is implicit;  $E_m, E_n, E_k$  are zeroth-order energy levels; all matrix elements and their derivatives are taken at  $E = E_n$ . It may be easily shown that for calculation of  $\Delta\nu$  to order  $m\alpha^6$  it is enough to use only three first terms in (9). Keeping only terms of the order desired, the recoil contribution to  $\Delta\nu$  equals (up to the order  $m\alpha^6$ )

$$\langle \psi | V_1 G' V_1 G' V_1 + V_2 G' V_1 + V_1 G' V_2 + V_3 + V_1 G' V_1 + V_2 + V_1 | \psi \rangle \Big|_{\sigma=0}^{\sigma=1}, \quad (10)$$

where  $|\psi\rangle$  is zeroth approximation to  $|e\rangle$ , i.e. solution of equation

$$H^{(0)} |\psi\rangle = E_2 |\psi\rangle, \quad (11)$$

and  $G' = G'(E_2)$  is Green function of equation (11) with the ground state pole subtracted out (the Green function without the subtraction is denoted as  $G$ ).

As  $H^{(0)}$  does not depend on fermion spins,  $|\psi\rangle$  may be assumed to have the form

$$|\psi\rangle = \int \frac{d^3 p}{(2\pi)^3} \varphi(\mathbf{p}) |\mathbf{p}\rangle |\chi\rangle,$$

where  $|\chi\rangle$  is spin part of the wave function. So to find  $|\psi\rangle$  is essentially to find  $\varphi(\mathbf{p})$ , i.e. to solve equation

$$(T + V_0 - E_2) \varphi(\mathbf{p}) = 0 \quad (12)$$

where  $T$  is kinetic energy of the two particles (including the mass).

Nonrelativistic approximation for (12) is

$$(T_0 + V_0 - E_0) \varphi_0(\mathbf{p}) = 0, \quad (13)$$

where  $T_0$  is nonrelativistic approximation for  $T$ . The ground state solution for (13) is

$$E_0 = 2m - \frac{\gamma^2}{m}, \quad \varphi_0(\mathbf{p}) = 8 \gamma^{5/2} \pi^{1/2} \frac{1}{f_p^2}$$

(here and below notations  $\gamma \equiv \alpha m/2$ ,  $f_k \equiv \mathbf{k}^2 + \gamma^2$  are used, for an arbitrary momentum  $\mathbf{k}$ ).

Let us write  $\varphi$  as

$$\varphi = C_0 \varphi_0 + \Delta\varphi ,$$

where  $\Delta\varphi$  is orthogonal to  $\varphi_0$ ; then (12) is rewritten as

$$(T_0 + V_0 - E_0) \Delta\varphi = (-\Delta T + \Delta E) \varphi , \quad (14)$$

where  $\Delta E \equiv E_2 - E_0$  ,  $\Delta T \equiv T - T_0$  . Solution of (14) may be written in the form

$$\Delta\varphi = G'_0(E_0) (\Delta T - \Delta E) \varphi , \quad (15)$$

where  $G'_0$  is Green function of equation (13) (i.e. usual nonrelativistic Coulomb Green function) with the ground state pole subtracted out (corresponding Green function without the subtraction is denoted as  $G_0$ ).

Relativistic free two-particle Green function and nonrelativistic free one are referred to as  $S_0$  and  $S$ :

$$\begin{aligned} S_0(E') &= (E' - T_0)^{-1} , \\ S(E') &= (E' - T)^{-1} ; \end{aligned}$$

$G$  and  $G_0$  are expressed as expansions

$$\begin{aligned} G_0(E') &= (E' - (T_0 + V_0))^{-1} = S_0 + S_0 V_0 S_0 + \dots , \\ G(E') &= (E' - (T + V_0))^{-1} = S + S V_0 S + \dots . \end{aligned} \quad (16)$$

Let us also denote

$$L \equiv G'_0 - S_0 - S_0 V_0 S_0 .$$

It may be shown by iteration of (15) that with the sufficient accuracy its solution is

$$\varphi \approx (1 + (S - S_0)V_0 + S V_0(S - S_0)V_0 + L \Delta T) |_{E'=E_0} \varphi_0 . \quad (17)$$

It is convenient to use in the rest of the paper notations  $\langle \rangle$  and  $\langle \rangle_{(n)}$  , defined so that for any operator  $X$

$$\langle X \rangle \equiv \int \varphi_0^+(\mathbf{p}') X(\mathbf{p}, \mathbf{p}', E) |_{\sigma=0}^{\sigma=1} \varphi_0(\mathbf{p}) \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} ,$$

and  $\langle X \rangle_{(n)}$  stands for the sum of contributions  $\sim \alpha^n, \alpha^n \ln \alpha$  to  $\langle X \rangle$ .

If one rewrites (10) using

$$G' \approx (S + S V_0 S + L) |_{E'=E_0} ,$$

expression for  $\varphi$  from  $\varphi_0$  (17), and keeping only terms of the relevant order, one obtains

$$\Delta\nu_{rec} = \langle V_3 + V_2 + U_{V2} + U_C + U_M + U_{MM} + U_{MCM} + U_{MMM} + U_L \rangle_{(6)} , \quad (18)$$

where

$$U_{V2} = (V_2 S V_m + V_2 (S V_c - S_0 V_0)) + h.c. , \quad (19)$$

$$U_C = V_{1c} + U_{C2} + U_{C3} , \quad (20)$$

$$U_M = V_m + U_{M2} + U_{M3} , \quad (21)$$

$$U_{MM} = U_{MM2} + U_{MM3} \quad (22)$$

$$U_{MCM} = V_m S V_c S V_m , \quad (23)$$

$$U_{MMM} = V_m S V_m S V_m , \quad (24)$$

$$U_L = V_1 L V_1 + ( V_1 L \Delta T + h.c. ) , \quad (25)$$

and

$$U_{C2} = V_{1c} S V_{1c} + ( V_{1c} (S - S_0) V_0 + h.c. ) , \quad (26)$$

$$U_{M2} = V_m (S V_c - S_0 V_0) + h.c. , \quad (27)$$

$$U_{MM2} = V_m S V_m , \quad (28)$$

$$U_{C3} = V_0 (S - S_0) V_{1c} (S - S_0) V_0 + ( V_{1c} S V_0 (S - S_0) V_0 + h.c. ) + \quad (29)$$

$$+ ( V_{1c} S V_{1c} (S - S_0) V_0 + h.c. ) + V_{1c} S V_0 S V_{1c} + V_{1c} S V_{1c} S V_{1c} ,$$

$$U_{M3} = (V_c S - V_0 S_0) V_m (S V_c - S_0 V_0) + ( V_m S V_c (S V_c - S_0 V_0) + h.c. ) , \quad (30)$$

$$U_{MM3} = ( V_m S V_m (S V_c - S_0 V_0) + h.c. ) . \quad (31)$$

### 3. THE METHOD OF THE CALCULATION.

In this section method of calculation of contributions to  $\Delta\nu_{rec}$ , written out in (18) - (31), is presented. The main idea of this method is just the idea used in other modern works concerning bound states, though there is some difference in its implementation. Every contribution to  $\Delta\nu_{rec}$  is divided essentially into two parts, "soft" and "hard" ones. Here the following is undermined. Contributions to  $\Delta\nu_{rec}$  are in fact integrals over loop momenta. Any contribution under the consideration is called "hard" if it is determined (with accuracy required) by region where all the loop momenta are of order  $m$ . Otherwise this contribution is called "soft". To separate "soft" contributions from "hard" ones, the integrands are to be expanded in powers of the momenta. In order to know whether given term of such expansion contain "soft" contribution, all the momenta should be set to  $\sim \alpha m$ , and then simplest power counting let us know whether this momentum area give rise contribution  $\sim m\alpha^6$  to  $\Delta\nu_{rec}$  (for the term under the consideration). If this is the case, the term considered clearly must be treated as "soft" contribution. On the other hand it can be proved that in the present problem absence of the contribution  $\sim m\alpha^6$  due to the momenta region mentioned above lead to absence of "soft" contributions of the relevant order at all, at least for the way of calculation described in the present paper.

It is clear that there is some freedom in explicit way of the decomposition into "soft" and "hard" parts, and "soft" part can always be defined so that its contribution to the integral is easily evaluated analytically. On the other hand, calculating "hard" part one can put  $E_0 = 2m$  in the integrands, and the integrals obtained can be easily evaluated by means of numerical integration.



Contributions to  $\Delta\nu_{rec}$  are naturally divided into three sets, different in form and calculation procedure used. These are "tree" contributions ( $< V_{1c}, V_m >_{(6)}$ ), "one-loop" ( $< V_2, U_{C2}, U_{M2}, U_{MM2} >_{(6)}$ ), and "two-loop" ones ( $< V_3, U_{V2}, U_{C3}, U_{M3}, U_{MM3}, U_{MCM}, U_{MMM} >_{(6)}$ ) (here words "one-" and "two-loop" imply that the operators inside  $< >_{(6)}$  are integrals over one and two momenta respectively), and also term  $< U_L >_{(6)}$  that corresponds to graphs having three loops or more; the last is determined by region of momenta  $\sim m\alpha$  and can be easily found analytically.

Consider different kinds of the contributions one by one.

Each two-loop contribution has the form  $< X_2 >_{(6)}$  where

$$X_2(\mathbf{p}, \mathbf{p}', \gamma) = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \alpha^3 Y_2(\mathbf{p}, \mathbf{p}', \mathbf{q}_1, \mathbf{q}_2, \gamma) . \quad (32)$$

It is convenient to define the value  $n$ , the "divergency power" of the  $X_2$ ; namely,  $Y_2 \sim \delta^{n-6}$  at  $p \sim p' \sim q_1 \sim q_2 \sim \gamma \sim \delta \ll m$ . All the "two-loop" contributions have  $n \geq 0$ ; if  $n = 0$  then region  $q_1 \sim q_2 \sim m\alpha$  contribute to order  $\sim m\alpha^6$  and  $< X_2 >_{(6)}$  may contain terms  $\sim m\alpha^6 \ln \alpha$ . It can be shown that for all the "two-loop" terms having  $n > 0$  the only essential contribution to the integral arises at  $p \sim p' \sim \gamma$ ,  $q_1 \sim q_2 \sim |\mathbf{q}_1 - \mathbf{q}_2| \sim m$ ; dependence of  $Y_2$  on  $\mathbf{p}, \mathbf{p}'$  in this area can be neglected that results in

$$< X_2 >_{(6)} \approx |\varphi_0(0)|^2 X_2|_{\mathbf{p}=\mathbf{p}'=\gamma=0}^{\sigma=1}_{\sigma=0} \equiv < X_2 >_p , \quad (33)$$

where

$$\varphi_0(0) = \int \frac{d^3 p}{(2\pi)^3} \varphi_0(\mathbf{p}) = \sqrt{\frac{\gamma^3}{\pi}} .$$

The integral in (33) does not depend on the small parameter  $\alpha$  and so can be easily computed numerically. For the sake of brevity notation  $< >_p$  defined in (33) is used throughout the paper.

If  $n = 0$  the integrand  $Y_2$  is to be divided into two parts: the soft part, denoted as  $Y_{21}$ , and the hard one,  $Y_{22}$ ,  $Y_2 = Y_{21} + Y_{22}$ , so that  $Y_{21}, Y_{22}$  have "divergency power"  $n = 0$  and  $n > 0$  respectively. It may be done in such a way that  $Y_{21}$  equals approximately a uniform function at  $p, p', q_1, q_2, \gamma \ll m$  (the only exception is the contribution of graph shown at fig.2(b), treated in the section 4). Similarly  $X_2 = X_{21} + X_{22}$  where  $X_{21}, X_{22}$  are integrals of form (32) with the  $Y_2$  replaced by  $Y_{21}, Y_{22}$  respectively. It may be shown that with such choice of  $Y_{21}$  in all cases met in calculations below the correct recipe for the calculation is

$$< X_2 >_{(6)} = < X_2 - X_{21} >_p + < X_{21} >_{(6)} . \quad (34)$$

The decomposition into  $X_{21}$  and  $X_{22}$  is evidently non-unique, and  $X_{21}$  may be chosen so as readily to calculate  $< X_{21} >_{(6)}$  analytically.

In the method described above the  $< X_{21} >_{(6)}$  is defined by the integral which is assumed to converge. Hence the  $Y_{21}$  falls at  $q_1, q_2 \gg m$  faster than uniform function having  $n = 0$ . As the scale of the momenta, at which the original function  $Y_2$  begins to fall rapidly, is  $q_1 \sim q_2 \sim m$ , it is natural to demand the same behavior from the function  $Y_{21}$  as well; in

other words, there exist effective "cutoff" of the  $Y_{21}$  at  $q_1 \sim q_2 \sim m$ . This "cutoff" reminds regularizations in nonrelativistic effective theories; however, in the approach described here there is no need in artificial regulators such as large regulating mass or dimension different from 4. Similarly to what explained above, in the rest of the paper all functions looking like uniform (with  $n = 0$ ) at small momenta fall rapidly enough at large ones.

For calculation of "tree" and "one-loop" contributions it is necessary to find corrections to "leading" terms, which are of first and second relative order in the  $\alpha$ . The method of finding of these corrections is rather obvious and is shown by the following simple illustration. Consider calculation of  $\langle X \rangle$  where  $X = X(\mathbf{p}, \mathbf{p}')$  is sufficiently smooth function that doesn't depend on  $\alpha$ , and expansion of  $X(\mathbf{p}, \mathbf{p}') - X(0, 0)$  in  $\mathbf{p}, \mathbf{p}'$  starts from third-order terms. Evidently

$$\langle X \rangle = (A_0 + A_1 + A_2)|_{\sigma=0}^{\sigma=1} ,$$

$$A_0 = |\varphi_0(0)|^2 X(0, 0) ,$$

$$A_1 = \varphi_0(0) \int \frac{d^3 p}{(2\pi)^3} \varphi_0(\mathbf{p}) (X(\mathbf{p}, 0) + X(0, \mathbf{p}) - 2X(0, 0)) , \quad (35)$$

$$A_2 = \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \varphi_0^+(\mathbf{p}') (X(\mathbf{p}, \mathbf{p}') - X(\mathbf{p}, 0) - X(0, \mathbf{p}') + X(0, 0)) \varphi_0(\mathbf{p}) . \quad (36)$$

It is easily seen that when calculating three first terms of expansion of  $\langle X \rangle$  in  $\alpha$ , in (35,36)  $\varphi_0(\mathbf{p})$  may be replaced by  $8\gamma^{5/2}\pi^{1/2} p^{-4}$ ; hence  $A_{0,1,2}$  exactly have orders  $\alpha^3$ ,  $\alpha^4$ ,  $\alpha^5$  respectively.

Each "one-loop" contribution has the form  $\langle X_1 \rangle_{(6)}$  where

$$X_1(\mathbf{p}, \mathbf{p}', \gamma) = \int \frac{d^3 q}{(2\pi)^3} \alpha^2 Y_1(\mathbf{p}, \mathbf{p}', \mathbf{q}, \gamma) ; \quad (37)$$

Let  $n$  to be defined so that  $Y_1 \sim \delta^{n-2}$  at  $p \sim p' \sim q \sim \gamma \sim \delta \ll m$  (if one writes  $X_1 S_0 V_0$  as the loop integral, the  $n$  is its "divergency power";  $n = 0$  imply that the region  $q \sim m\alpha$  contributes to order  $\sim m\alpha^6$ , and may cause terms  $\sim m\alpha^6 \ln \alpha$ ) to exist.

Let us write  $Y_1$  for any given contribution as sum of "soft" and "hard" parts,  $Y_1 = Y_{11} + Y_{12}$ , where  $Y_{11}$  and  $Y_{12}$  has  $n \leq 0$  and  $n > 0$  respectively (similarly  $X_1 = X_{11} + X_{12}$ ). It may be done so that  $Y_{11}$  at  $p, p', q, \gamma \ll m$  equals approximately a uniform function (the only exception is the contribution of graph shown at fig.2(a), which is treated in the section 4). It can be proved that under such choice of  $Y_{11}$  contributions coming from  $Y_{12}$  can be obtained, for all terms to treat, using formula like (35), which results in

$$\langle X_1 \rangle_{(6)} = \langle ((X_1 - X_{11}) - (X_1 - X_{11})_0) S_0 V_0 + h.c. \rangle_p + \langle X_{11} \rangle_{(6)} ; \quad (38)$$

in (38) the notation  $(\ )_0$ , is used, defined so that  $(X)_0(\mathbf{p}, \mathbf{p}') \equiv X(\mathbf{p}, \mathbf{p}')|_{\mathbf{p}=\mathbf{p}'=0}$ . Evidently  $Y_{11}$  can always be chosen so that second term in (38) is easily evaluated analytically.

"Tree" contributions to  $\Delta\nu_{rec}$  are calculated in the way just like one described above. There are only two "tree" contributions to be found:  $\langle V_m \rangle_{(6)}$  and  $\langle V_{1c} \rangle_{(6)}$ . Each of them is divided into "soft" and "hard" parts (the explicit way of this partition is described in the section 4), and then the "hard" contributions are evaluated according to (36).

In the section 4 separation of "soft" contributions from  $\Delta\nu_{rec}$  is done using formulas (34),(38),(36), that results in

$$\langle V_3 \rangle_{(6)} = \langle V_3 - W_{V3} \rangle_p + E_{V3}^S, \quad (39)$$

$$\langle V_2 + U_{V2} \rangle_{(6)} = \langle V_2 SV_m + h.c. \rangle_p + \langle (V_2 SV_c - W_{V2}) + h.c. \rangle_p + E_{V2}^S, \quad (40)$$

$$\langle U_C \rangle_{(6)} = \langle V_c SV_c SV_c - W_C \rangle_p + E_C^S, \quad (41)$$

$$\langle U_M \rangle_{(6)} = \langle V_c SV_m SV_c - W_{M1} \rangle_p + \langle (V_m SV_c SV_c - W_{M2}) + h.c. \rangle_p + E_M^S, \quad (42)$$

$$\langle U_{MM} \rangle_{(6)} = \langle (V_m SV_m SV_c - W_{MM}) + h.c. \rangle_p + E_{MM}^S, \quad (43)$$

$$\langle U_{MCM} \rangle_{(6)} = \langle V_m SV_c SV_m - W_{MCM} \rangle_p + E_{MCM}^S, \quad (44)$$

$$\langle U_{MMM} \rangle_{(6)} = \langle V_m SV_m SV_m \rangle_p, \quad (45)$$

here  $W_{V3}$ ,  $W_{V2}$ ,  $W_C$ ,  $W_{M1}$ ,  $W_{M2}$ ,  $W_{MM}$ ,  $W_{MCM}$  are operators associated with the soft contributions, which have relatively simple form,  $E_{V3}^S$ ,  $E_{V2}^S$ ,  $E_M^S$ ,  $E_{MM}^S$ ,  $E_{MCM}^S$ ,  $E_C^S$  are analytically found values of "soft" contributions to  $\Delta\nu_{rec}$ ; these values and operators are written explicitly in the section 4.

Remind that operators  $V_0 + V_1$ ,  $V_2$ ,  $V_3$  are given by sums of all the tree, one-loop, and two-loop irreducible noncovariant graphs respectively. Hence first terms in  $\langle \rangle_p$  in (39) - (45) are also given by sums of some noncovariant graphs (reducible ones, except  $V_3$ ). The sum of them is equal to the sum of expressions associated with all two-loop graphs of the recoil type. The operator corresponding to this sum is referred to as  $W$ . For convenience of references, the following notations are also used: sum of all second terms in  $\langle \rangle_p$  in (39) - (45) is denoted as  $W_0$ , and sum of all the soft contributions as  $E^S$ , hence

$$\Delta\nu_{rec} = \langle W - W_0 \rangle_p + E^S;$$

also, notations  $W'$ ,  $W'_0$  are used, defined according to

$$W, W_0(\mathbf{p}, \mathbf{p}', \gamma)|_{\mathbf{p}=\mathbf{p}'=\gamma=0} = \int \frac{d^3 p_1}{(2\pi)^3} \frac{d^3 p_2}{(2\pi)^3} W', W'_0(\mathbf{p}_1, \mathbf{p}_2).$$

A sum of contributions to  $W'$ , arising from all noncovariant graphs with the same topological structure (taking into account the difference of magnetic and Coulomb quanta), is equal to integrand associated with usual covariant Feynman graph (after the integration over zero components of the loop momenta). Hence the  $W'$  can be determined as sum of contributions of all *covariant* two-loop graphs of the recoil type, and there is no need to know contributions to the  $W'$  arising from separate noncovariant graphs.

In the above the gauge used for noncovariant and covariant graphs was undermined to be the same. However sum of contributions of all covariant graphs to the  $W'$  is independent of gauge, and so in actual calculations any gauge may be choosen. This is quite natural; in terms of the NRQED (or the effective nonrelativistic Hamiltonian approach)  $\langle W - W_0 \rangle_p$  is equal, besides some overall factor, to contribution  $\sim \alpha^3$  to the constant of pointlike

fermion-fermion interaction (if an appropriate regularization is used), which is determined from scattering amplitudes on the mass shell.

#### 4. THE "SOFT" CONTRIBUTIONS.

In this section the general method of separation of "soft" contributions, described above, is applied to various terms of (18) - (31) and to contributions of various graphs.

There are some preliminary remarks concerning the calculation technique employed. First, every term contributing to  $W$  and  $W_0$  evidently may be replaced by its average over positronium polarizations (or, in other words, over directions of the full spin). This simplifies the calculations considerably. For contributions to  $W$  this average is found in program way, as  $1/3$  of sum over polarizations. For contributions to  $W_0$  the averaging is performed using the following formulas (to the r.h.s. of which terms independent of the full spin may be added):

$$(\vec{\sigma}_1 \mathbf{a}) (\vec{\sigma}_2 \mathbf{b}) \rightarrow \frac{1}{3} (\vec{\sigma}_1 \vec{\sigma}_2) (\mathbf{a} \mathbf{b}) , \quad (46)$$

$$(\vec{\sigma}_1 \mathbf{a}) (\vec{\sigma}_2 \mathbf{b}) \left( \vec{\sigma}_1 \vec{\sigma}_2 - \frac{1}{\mathbf{q}^2} (\vec{\sigma}_1 \mathbf{q}) (\vec{\sigma}_2 \mathbf{q}) \right) \rightarrow -\frac{1}{3} \vec{\sigma}_1 \vec{\sigma}_2 (\mathbf{a} \mathbf{b} + (\mathbf{a} \mathbf{q})(\mathbf{b} \mathbf{q})/\mathbf{q}^2) , \quad (47)$$

$$\left( \vec{\sigma}_1 \vec{\sigma}_2 - \frac{(\vec{\sigma}_1 \mathbf{q}) (\vec{\sigma}_2 \mathbf{q})}{\mathbf{q}^2} \right) \left( \vec{\sigma}_1 \vec{\sigma}_2 - \frac{(\vec{\sigma}_1 \mathbf{k}) (\vec{\sigma}_2 \mathbf{k})}{\mathbf{k}^2} \right) \rightarrow \vec{\sigma}_1 \vec{\sigma}_2 \left( -1 + \frac{1}{3} \frac{(\mathbf{q} \mathbf{k})^2}{\mathbf{q}^2 \mathbf{k}^2} \right) . \quad (48)$$

Second, let me describe calculation of contributions associated with retardation. These contributions have "ultrasoft" parts, for which virtual photon momenta  $\sim m\alpha^2$  are essential. These contributions are contained in  $\langle V_m \rangle_{(6)}, \langle V_2 \rangle_{(6)}, \langle V_3 \rangle_{(6)}$ . Considering them, it is convenient to replace these operators by their spin-spin parts averaged over positronium polarizations, as described above. First nonrelativistic approximations (in the sense of the power expansion in momenta) for averaged contributions of graphs 2(a) and 2(b) to  $V_2$  and  $V_3$  are  $V_2^{ret}$  and  $V_3^{ret}$ ,

$$V_2^{ret}(\mathbf{p}, \mathbf{k}) = \int \frac{d^3 q}{(2\pi)^3} \alpha^2 \frac{4}{3} \frac{\pi^2}{m^2} \vec{\sigma}_1 \vec{\sigma}_2 \frac{q}{q'^2} Y_1^{ret}(\mathbf{p}, \mathbf{k}, \mathbf{q}) , \quad (49)$$

$$V_3^{ret}(\mathbf{p}, \mathbf{k}) = \int \frac{d^3 q_1}{(2\pi)^3} \frac{d^3 q_2}{(2\pi)^3} \alpha^3 \frac{16}{3} \frac{\pi^3}{m^2} \vec{\sigma}_1 \vec{\sigma}_2 \frac{q}{q_1^2 q_2^2} Y_2^{ret}(\mathbf{p}, \mathbf{k}, \mathbf{q}_1, \mathbf{q}_2) , \quad (50)$$

where

$$Y_1^{ret}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \frac{1}{q + (f_p + f_{k_1})/2} \frac{1}{q + (f_k + f_{p_1})/2} , \quad (51)$$

$$Y_2^{ret}(\mathbf{p}, \mathbf{k}, \mathbf{q}_1, \mathbf{q}_2) = \frac{1}{(q + (f_p + f_{k_2})/2)} \frac{1}{(q + (f_{p_1} + f_{k_1})/2)} \frac{1}{(q + (f_{p_2} + f_k)/2)} \quad (52)$$

(notations for the momenta are shown at fig. 2); just the same contributions arise due to graphs obtained from graphs 2(a) and 2(b) by the time reversal.

Spin-spin part of  $V_m$ , averaged over positronium polarizations, is denoted below as  $V_s$ . First nonrelativistic approximation to  $V_s(\mathbf{p}, \mathbf{k})$  is

$$V_m^{ret}(\mathbf{p}, \mathbf{k}) = V_{m0} \frac{q}{q + (f_p + f_k)/2} , \quad (53)$$

where

$$V_{m0} = V_s|_{\mathbf{p}=\mathbf{k}=\gamma=0} = \frac{2}{3} \frac{\alpha\pi}{m^2} \vec{\sigma}_1 \vec{\sigma}_2 , \quad \mathbf{q} \equiv \mathbf{k} - \mathbf{p} .$$

It is convenient to consider  $\langle V_m^{ret} \rangle_{(6)}, \langle V_2^{ret} \rangle_{(6)}, \langle V_3^{ret} \rangle_{(6)}$  together and to calculate their overall contribution to  $\Delta\nu_{rec}$ ; this way of the calculation was employed, in particular, in [1] and [2]. The way of the calculation, used in the present paper is, in some sense, even shorter. Each of operators  $V_m^{ret}, V_2^{ret}, V_3^{ret}$  is divided into two parts:

$$\begin{aligned} V_m^{ret}(\mathbf{p}, \mathbf{k}) &= V_{m0}^{ret}(\mathbf{p}, \mathbf{k}) + V'_{m0}(\mathbf{p}, \mathbf{k}) , \\ V_2^{ret}(\mathbf{p}, \mathbf{k}) &= V_{20}^{ret}(\mathbf{p}, \mathbf{k}) + V'_{20}(\mathbf{p}, \mathbf{k}) , \\ V_3^{ret}(\mathbf{p}, \mathbf{k}) &= V_{30}^{ret}(\mathbf{p}, \mathbf{k}) + V'_{30}(\mathbf{p}, \mathbf{k}) , \end{aligned}$$

where

$$V_{m0}^{ret}(\mathbf{p}, \mathbf{k}) = \frac{1}{2} V_{m0} \left[ \frac{q}{q + f_p} + \frac{q}{q + f_k} \right] , \quad (54)$$

and  $V_{20}^{ret}, V_{30}^{ret}$  are obtained from  $V_2^{ret}, V_3^{ret}$  by replacing  $Y_1^{ret}, Y_2^{ret} \rightarrow Y_{10}^{ret}, Y_{20}^{ret}$  where

$$Y_{10}^{ret}(\mathbf{p}, \mathbf{k}, \mathbf{q}) = \frac{1}{2} \left( \frac{1}{(q + f_k)(q + f_{k_1})} + \frac{1}{(q + f_p)(q + f_{p_1})} \right) , \quad (55)$$

$$Y_{20}^{ret}(\mathbf{p}, \mathbf{k}, \mathbf{q}_1, \mathbf{q}_2) = \frac{1}{2} \left( \frac{1}{(q + f_p)(q + f_{p_1})(q + f_{p_2})} + \frac{1}{(q + f_k)(q + f_{k_1})(q + f_{k_2})} \right) . \quad (56)$$

In the region where all the momenta are  $\sim m\alpha$

$$V_m^{ret} \approx V_{m0}^{ret} , \quad Y_1^{ret} \approx Y_{10}^{ret} , \quad Y_2^{ret} \approx Y_{20}^{ret} .$$

If one writes  $V'_{20}, V'_{30}$  as integrals like (49,50), the integrands in them have powers, being expanded in the momenta, greater by 2 than those of  $Y_1^{ret}, Y_2^{ret}$ ; also  $V'_{m0}$  has power, being expanded in the momenta, greater by 2 than the power of  $V_m^{ret}$ . Due to this reason  $\langle V'_{20} \rangle_{(6)}, \langle V'_{30} \rangle_{(6)}$  consist of "hard" contributions only. As for  $\langle V'_{m0} \rangle_{(6)}$ , it does contain soft contributions. However it may be easily calculated in just the way other soft contributions to  $\langle V_m \rangle_{(6)}$  are calculated, and is actually evaluated combined with them in sec.4.3.

Now turn to contributions of  $V_{m0}^{ret}, V_{20}^{ret}, V_{30}^{ret}$ . It is easy to see that their full contribution is equal to 0:

$$\langle V_{m0}^{ret} \rangle_{(6)} + 2 \langle V_{20}^{ret} \rangle_{(6)} + 2 \langle V_{30}^{ret} \rangle_{(6)} = 0 . \quad (57)$$

This may be shown by a short direct calculation; however (57) is evident immediately due to the following reason. Consider calculation of the recoil contribution  $\sim \alpha^6 m^2/M$  to

ground state hyperfine splitting of the "hydrogen", i.e. particle of mass  $m/2$  moving around particle of large mass  $M$ . In this calculation  $V_m^{ret}(\mathbf{p}, \mathbf{k})$ , for instance, must be replaced by  $2m/M V_{m0}^{ret}(\mathbf{p}, \mathbf{k})$ ; entirely  $\langle V_m^{ret} + 2V_2^{ret} + 2V_3^{ret} \rangle_{(6)}$  is replaced by  $2m/M \langle V_{m0}^{ret} + 2V_{20}^{ret} + 2V_{30}^{ret} \rangle_{(6)}$ , as may be easily seen. On the other hand, it is well known that when calculating hyperfine splitting in first order in  $m/M$  the magnetic interaction may be regarded as instant, which immediately leads to

$$\langle V_{m0}^{ret} + 2V_{20}^{ret} + 2V_{30}^{ret} \rangle_{(6)} = \langle V_{m0} \rangle_{(6)} = 0 . \quad (58)$$

The last point to discuss is calculation of  $\langle U_L \rangle_{(6)}$ , that is essentially the contribution of graphs containing Coulomb ladder with three or more loops. This contribution to  $\Delta\nu_{rec}$  arises completely, with the accuracy required, from the region where all the momenta in the loops are of order  $m\alpha$ , and can be easily evaluated analytically. There are two kinds of such contributions, namely, graphs having one or two magnetic quanta. The contribution of graphs having two magnetic quanta is

$$E_{LMM}^S \equiv \langle V_m L V_m \rangle_{(6)}$$

and does not depend, in the order required, on details of formalism used; for the first time it has been found in [26]. Method of its evaluation, used in the present paper, coincides mainly with the method used in [27] (and involves evaluation in coordinate representation); the result obtained agrees with those found earlier and equals

$$E_{LMM}^S = \left( \frac{791}{864} - \frac{\pi^2}{18} \right) m\alpha^6 \approx 0.3672 m\alpha^6 . \quad (59)$$

The contribution of graphs having one magnetic photon does depend on details of formalism used. The value calculated in the present paper is

$$E_{LM}^S \equiv \langle (V_m L (V_{1c} + \Delta T)) + h.c. \rangle_{(6)} .$$

This value was evaluated in two ways: in coordinate space (by method close to that of [27]), and by immediate integration over the momenta, using explicit form of  $L(\mathbf{p}, \mathbf{k})$ , quoted, for instance, in [26]:

$$L(\mathbf{p}, \mathbf{k}) = \frac{-4\pi m^4 \alpha^3}{f_p^2 f_k^2} \left( \frac{5}{2} - 4\frac{\gamma^2}{f_p} - 4\frac{\gamma^2}{f_k} + \frac{1}{2} \ln A + \frac{2A-1}{\sqrt{4A-1}} \arctan \sqrt{4A-1} \right) , \quad (60)$$

where

$$A = \frac{f_p f_k}{4\gamma^2 q^2} .$$

The results obtained by both methods are the same and equal to

$$E_{LM}^S = \frac{1}{64} m\alpha^6 \approx 0.01562 m\alpha^6 . \quad (61)$$

Finally, addition (59) and (61) together results in

$$\langle U_L \rangle_{(6)} \approx 0.3728 m\alpha^6 .$$

#### 4.1. Irreducible Graphs.

For calculation of  $\langle V_3 \rangle_{(6)}$  it is necessary first to separate from  $V_3$  the term  $V_{30}$  which is an integral having the "divergency power"  $n = 0$ , and to use formula (34) where  $X_{21}$  should be set equal to  $V_{30}$ ; the result has the form (39) with

$$W_{V_3} = V_{30} , \quad E_{V_3}^S = \langle V_{30} \rangle_{(6)} .$$

There exist only three essentially different graphs contributing to  $V_{30}$ : these are graphs depicted at fig. 3(a), 3(b) and 2(b). However contribution to  $V_{30}$ , due to sum of graphs 3(a) and 3(b), may be set equal to zero. The matter is that this sum in the region of small loop momenta reduces effectively to the graph 3(c), in which effective two-photon "seagull" vertex does not depend on the spins. This vertex appears as sum of subgraphs 3(d) and 3(e) and corresponds to term  $\alpha \mathbf{A}^2 / (2m)$  (where  $\mathbf{A}$  is the vector potential) in the nonrelativistic Hamiltonian of particle in magnetic field. Thus  $V_{30}$  arises from the only graph 2(b) and may be chosen to be  $2V_{30}^{ret}$  which results in  $E_{V_3}^S = 2 \langle V_{30}^{ret} \rangle_{(6)}$ ; in fact there is no need to evaluate this value, as mentioned above.

Similarly  $\langle V_2 \rangle_{(6)}$  is calculated using (38) with  $X_{11} = V_{20}$  where  $V_{20}$  is part of  $V_2$ , having  $n = 0$ . Noting that  $\langle U_{V_2} \rangle_{(6)} = \langle U_{V_2} \rangle_p$ , as is easily seen, one finds (40) where

$$W_{V_2} = (V_{20} + (V_2 - V_{20})_0) S_0 V_0 , \quad E_{V_2}^S = \langle V_{20} \rangle_{(6)} .$$

Consider contributions to  $V_{20}$ . Graphs having two Coulomb quanta do not contribute to  $V_{20}$ . Among graphs having two magnetic quanta there are those with  $n = -1$  (graphs depicted in fig. 3(f), (g)), and several graphs with  $n = 0$ . However their contributions to  $V_{20}$  cancel each other, just as described above for  $V_{30}$ ; for instance, sum of graphs shown in fig. 3(f) and 3(g) reduces effectively in the region of small momenta to the graph 3(h), in which effective seagull vertex does not depend on spins. Graphs having one magnetic and one Coulomb quanta, contributing to  $V_2$ , are divided into 4 sets belonging each to one of covariant graphs (A), (B), shown in fig. 4, and the graphs obtained by the time reversal. It is convenient to consider graphs including electron-positron pairs and those without pairs separately.

Contributions of graphs including pairs to  $V_{20}$  may be chosen to be

$$V_{20}^A(\mathbf{p}, \mathbf{p}') = V_{20}^B = \dots = \int \frac{d^3 q}{(2\pi)^3} \frac{2\alpha^2 \pi^2}{3m^3} \vec{\sigma}_1 \vec{\sigma}_2 \frac{1}{q^2 q'^2} ((q^2 + q'^2) R_q - k^2 R_k) , \quad (62)$$

(the momenta notations are shown in fig. 4,  $\mathbf{k} \equiv \mathbf{p}' - \mathbf{p}$ , and  $R_k \equiv m^2 / (m^2 + k^2)$  for an arbitrary  $\mathbf{k}$ ). It is easy to find their overall contribution to  $E_{V_2}^S$ , that equals

$$E_P^S = \frac{2}{3} m \alpha^6 \ln \alpha .$$

The only graphs without pairs, contributing to  $V_2$ , are the graph shown in fig. 2(a), discussed above, and the similar graph obtained from 2(a) by the time reversal. Their contribution to  $V_{20}$  is chosen to be  $2V_{20}^{ret}$ . So

$$E_{V_2}^S = E_P^S + 2 \langle V_{20}^{ret} \rangle_{(6)} . \quad (63)$$

The second term in (63) need not in fact be evaluated, as explained above.

#### 4.2. Graphs with one magnetic photon.

Consider calculation of  $\langle U_M \rangle_{(6)}$ , i.e. contribution to  $\Delta\nu_{rec}$  due to reducible graphs with one magnetic photon, except corresponding contribution to  $\langle U_L \rangle_{(6)}$ , considered above.

As discussed above, it is convenient to calculate  $\langle V_s \rangle_{(6)}$  instead of  $\langle V_m \rangle_{(6)}$ , where  $V_s$  is spin-spin part of  $V_m$ , averaged over positronium polarizations. Let us write  $V_s$  as an "expansion in powers of momenta":

$$V_s = V_{m0}^{ret} + V_{m1} + V_{m2} , \quad (64)$$

where  $V_{m0}^{ret}$  is defined in (54),  $V_{m1}(\mathbf{p}, \mathbf{k}, \gamma) \approx V_s - V_{m0}^{ret}$  at  $p \sim k \sim \gamma \ll m$ , and  $V_{m1}$  at  $p, k, \gamma \ll m$  equals approximately a uniform function of second power. It can be shown that  $\langle V_{m2} \rangle_{(6)}$  can be evaluated using (36) with  $X = V_{m2}$ . Actually there is no need to evaluate  $\langle V_{m0}^{ret} \rangle_{(6)}$ , as mentioned above. The value  $\langle V_{m1} \rangle_{(6)}$  can be easily found in analytic form, if one uses appropriate choice for  $V_{m1}$ . It is convenient to choose

$$V_{m1}(\mathbf{p}, \mathbf{k}) = \frac{1}{2m^2} V_{m0} \left[ \frac{p^2 k^2}{q^2} (R_p + R_k) - \frac{1}{q^2} (p^4 R_p + k^4 R_k) - (p^2 R_p + k^2 R_k) \right] , \quad (65)$$

where  $(\mathbf{q} \equiv \mathbf{p} - \mathbf{k})$ .

In the lowest order in momenta  $V_m \approx V_{mb}$  where  $V_{mb}$  is part of the Breit Hamiltonian due to magnetic photon exchange. The value  $\langle U_{M2} \rangle_{(6)}$  is calculated using (38) with  $X_{11} = V_{mb}(S_0 V'_{11} + S_{11} V_0) + h.c.$ , where  $S_{11}$ ,  $V'_{11}$  are "first terms of expansion" of  $S - S_0$  and  $V_c - V_0$  in the momenta (namely,  $V'_{11}(\mathbf{p}, \mathbf{k}) \approx V_c - V_0$ ,  $S_{11}(\mathbf{p}) \approx S - S_0$  at  $p \sim k \sim \gamma \ll m$ , and  $S_{11}$ ,  $V'_{11}$  at  $p, k \ll m$  equal approximately to uniform functions of zero power).

The value  $\langle U_{M3} \rangle_{(6)}$  is calculated using (34) with  $X_{21} = V_{mb} S_0 V_0 (S_0 V'_{12} + S_{12} V_0) + h.c.$ , where  $S_{12}(\mathbf{p})$ ,  $V'_{12}(\mathbf{p}, \mathbf{k}) \approx S_{11}(\mathbf{p})$ ,  $V'_{11}(\mathbf{p}, \mathbf{k})$  at  $p, k \ll m$  (but generally  $S_{12}, V'_{12} \neq S_{11}, V'_{11}$ ; the explicit way of the decomposition of  $S - S_0$  and  $V_c - V_0$  is determined by the purpose of convenience of subsequent calculations).

It is easy to see that in  $X_{11}$ ,  $X_{21}$  defined in such manner it is enough to take into account only spin-independent parts of  $V'_{11}, V'_{12}$  (denoted as  $V_{11}, V_{12}$  respectively) and spin-spin part of  $V_{mb}$ , that can be replaced by its average over positronium polarizations, i.e.  $V_{m0}$ . So  $\langle U_{M2} \rangle_{(6)}$  and  $\langle U_{M3} \rangle_{(6)}$  can be evaluated using  $X_{11} = V_{m0}(S_0 V_{11} + S_{11} V_0) + h.c.$ ,  $X_{21} = V_{m0} S_0 V_0 (S_0 V_{12} + S_{12} V_0) + h.c.$ .

It is convenient to choose

$$V_{12}(\mathbf{p}, \mathbf{k}) = \alpha\pi/m^2 R_p = \alpha\pi/m^2 (1 + p^2/m^2)^{-1} ;$$



$V_{11}$  is defined so that  $V_{11}(\mathbf{p}, \mathbf{k}) = V_{11}(\mathbf{p}, 0)$ ; under this condition an explicit form of  $V_{11}$  does not enter final expressions.

It is easy to find that  $S_{11}, S_{12}$  may be set to

$$S_{1i}(p) = S_{1i}(p)|_{\gamma=0} + \frac{1}{2} \frac{\gamma^2}{f_p} - \frac{1}{4} \frac{\gamma^4}{f_p^2}, \quad (i = 1, 2); \quad (66)$$

first term in (66) is choosen to be  $(S - S_0)|_{(\gamma=0)}$  for  $S_{11}$  and  $-R_p/4$  for  $S_{12}$ .

After a simple calculation  $\langle U_M \rangle_{(6)}$  is found to have the form (42) with

$$E_M^S = m\alpha^6 \left( -\frac{1}{3} \ln \alpha + \frac{53}{192} \right) + \langle V_{m0}^{ret} \rangle_{(6)},$$

$$W_{M1} = ((V_c S V_m)_0 S_0 V_0 + h.c.) + V_0 S_0 (V_{m0}^{ret} + V_{m1}) S_0 V_0 - ((V_0 S_0 (V_{m0}^{ret} + V_{m1}))_0 S_0 V_0 + h.c.),$$

$$W_{M2} = V_{m0} S_0 V_0 (S_0 V_{12} + S_{12} V_0) + (V_m S V_0 + (V_m S (V_c - V_0))_0) S_0 V_0$$

(when deriving  $W_{M2}$  the term

$$((V_m - V_{m0}) S_{11} V_0 - ((V_m - V_{m0}) S_{11} V_0)_0) S_0 V_0,$$

was added to it, which causes no additional contribution of order  $m\alpha^6$ ).

#### 4.3. Coulomb graphs.

The Coulomb contribution due to reducible graphs to  $\Delta\nu_{rec}$  is equal to  $\langle U_C \rangle_{(6)}$ . This value was found in the style described above:  $\langle V_{1c} \rangle_{(6)}$ ,  $\langle U_{C2} \rangle_{(6)}$ ,  $\langle U_{C3} \rangle_{(6)}$  (i.e. separate terms of  $\langle U_C \rangle_{(6)}$ ) were evaluated using (36,38,34) respectively. The calculation is simplified by the fact that in lowest (i.e. zeroth) order in momenta  $V_{1c}$  contains only spin-independent terms, and besides that

$$V_{1c}^S(\mathbf{p}, \mathbf{k})|_{\mathbf{p}=0} = V_{1c}^S(\mathbf{p}, \mathbf{k})|_{\mathbf{k}=0} = 0,$$

where  $V_{1c}^S$  is spin-spin part of  $V_{1c}$ .

It is easy to see that  $\langle U_C \rangle_{(6)}$  is determined by the formula (41), in which

$$W_C = V_0 S_0 V_{1c}' S_0 V_0, \quad E_C^S = \langle V_{1c}' \rangle_{(6)}; \quad (67)$$

here  $V_{1c}'$  is first nonvanishing term of "expansion" of  $V_{1c}^S$  in momenta (exactly,  $V_{1c}'(\mathbf{p}, \mathbf{k}) \approx V_{1c}^S(\mathbf{p}, \mathbf{k})$  at  $p \sim k \ll m$ , and  $V_{1c}'$  at  $p, k \ll m$  equals approximately a uniform function of second power).

In (67) one can replace  $V_{1c}'$  by its average over positronium polarizations (denoted as  $V_{1c}''$ ); it is convenient to set

$$V_{1c}''(\mathbf{p}, \mathbf{k}) = \frac{1}{24} \vec{\sigma}_1 \vec{\sigma}_2 \frac{\alpha\pi}{m^2} \left( -\frac{1}{2} \left( \frac{p^4}{q^2} - p^2 - \mathbf{p}\mathbf{k} \right) R_p + (\mathbf{p} \leftrightarrow \mathbf{k}) + \frac{p^2 k^2}{q^2} R_p \right);$$

using it, one easily obtains

$$E_C^S = -\frac{1}{48} m\alpha^6 \left( \ln \alpha + \frac{1}{4} \right) .$$

#### 4.4. Graphs with two magnetic quanta.

Turn to calculation of  $\langle U_{MM2} + U_{MM3} + U_{MCM} \rangle_{(6)}$ , i.e. contribution of reducible graphs with two magnetic quanta, except corresponding contribution to  $\langle U_L \rangle_{(6)}$ , considered above. The value  $\langle U_{MM2} \rangle_{(6)}$  is found using (38) with  $X_{11} = V_{mb} S_0 R V_{mb}$ , where  $R(\mathbf{p}, \mathbf{p}') = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}') R_p$ . Similarly, the value  $\langle U_{MCM} \rangle_{(6)}$  is found using (34) with  $X_{21} = V_{mb} S_0 V_0 S_0 R V_{mb}$ . It is easy to see that  $\langle U_{MM3} \rangle_{(6)} = \langle U_{MM3} \rangle_p$ . The results have form (43),(44) with

$$W_{MM} = V_{mb} S_0 R V_{mb} S_0 V_0 + (V_m S V_m - V_{mb} S_0 R V_{mb})_0 S_0 V_0 ,$$

$$W_{MCM} = V_{mb} S_0 V_0 S_0 R V_{mb} ,$$

$$E_{MM}^S = \langle V_{mb} S_0 R V_{mb} \rangle_{(6)} , \quad E_{MCM}^S = \langle V_{mb} S_0 V_0 S_0 R V_{mb} \rangle_{(6)} .$$

Spin-orbital part of  $V_{mb}$  does not contribute to  $E_{MM}^S$  and  $E_{MCM}^S$ . Spin-spin and spin-independent contributions to  $V_{mb}$  are denoted as  $V_{sb}$  and  $V_{lb}$  respectively; they are equal to

$$V_{sb}(\mathbf{p}, \mathbf{p}') = \frac{\alpha\pi}{m^2} \left( \vec{\sigma}_1 \vec{\sigma}_2 - \frac{1}{\mathbf{q}^2} (\vec{\sigma}_1 \mathbf{q}) (\vec{\sigma}_2 \mathbf{q}) \right) ,$$

$$V_{lb}(\mathbf{p}, \mathbf{p}') = -4 \frac{\alpha\pi}{m^2} \left( -\frac{(\mathbf{q}\mathbf{p})^2}{q^4} + \frac{p^2}{q^2} \right) , \quad \mathbf{q} \equiv \mathbf{p}' - \mathbf{p} .$$

As the values to calculate are average values over S-state,  $V_{lb}$  may be replaced by any function  $V'_{lb}$  such that

$$\int V_{lb}(\mathbf{p}, \mathbf{p}') d\vartheta_{\mathbf{p}\mathbf{p}'} = \int V'_{lb}(\mathbf{p}, \mathbf{p}') d\vartheta_{\mathbf{p}\mathbf{p}'} ;$$

It is convenient to choose

$$V'_{lb}(\mathbf{p}, \mathbf{p}') = -4 \frac{\alpha\pi}{m^2} \left( -\frac{1}{2} + \frac{1}{2} \left( \frac{p'^2}{q^2} + \frac{p^2}{q^2} \right) \right) .$$

For separate parts of  $E_{MM}^S$ ,  $E_{MCM}^S$  one obtains

$$\langle V'_{lb} S_0 R V_{sb} + V_{sb} S_0 R V'_{lb} \rangle_{(6)} = \left( -\frac{1}{3} \ln \alpha - \frac{1}{6} \right) m\alpha^6 ,$$

$$\langle V_{sb} S_0 R V_{sb} \rangle_{(6)} = \left( -\frac{1}{24} \ln \alpha - \frac{7}{96} \right) m\alpha^6 ,$$

$$\langle V'_{lb} S_0 V_0 S_0 R V_{sb} + V_{sb} S_0 V_0 S_0 R V'_{lb} \rangle_{(6)} = \left( -\frac{1}{6} + \frac{\pi^2}{18} \right) m\alpha^6 ,$$

$$< V_{sb} S_0 V_0 S_0 R V_{sb} >_{(6)} = \left( -\frac{5}{48} \ln \alpha + \frac{1}{96} \right) m\alpha^6 ,$$

which results in

$$E_{MM}^S = \left( -\frac{3}{8} \ln \alpha - \frac{23}{96} \right) m\alpha^6 ,$$

$$E_{MCM}^S = \left( -\frac{5}{48} \ln \alpha - \frac{5}{32} + \frac{\pi^2}{18} \right) m\alpha^6 .$$

Now the calculation of the "soft" contributions is completed. Their sum is

$$E^S = \left( -\frac{1}{6} \ln \alpha + \frac{1393}{1728} \right) m\alpha^6 \approx \left( -\frac{1}{6} \ln \alpha + 0.8061 \right) m\alpha^6 .$$

To check the formalism and the calculation technique used, they were applied to calculation of the recoil contribution  $\sim \alpha^6 m^2/M$  to the ground state hyperfine splitting of the hydrogen. It is well known that for this particular purpose the two-particle problem reduces effectively to the problem of motion in external field, and the value desired may be easily obtained by means of coordinate-space calculations, using known solutions of the Dirac equation in the Coulomb field. The correction to the hyperfine splitting thus obtained is equal to  $4\alpha^6 m^2/M$ . The same result was found using momenta representation by the method described in the present paper. To simplify corresponding calculations, diagram technique rules were modified, because in the relevant order one can consider the magnetic field as permanent, and so the magnetic photon exchange may be assumed to be instantaneous. All the "hard" contributions cancel each other; the soft contributions of the desired order arise only from graphs having one magnetic photon, and their calculation coincides mainly with the calculation of corresponding contributions for positronium.

## 5. THE "HARD" CONTRIBUTIONS.

For the calculation of the "hard" contribution to  $\Delta\nu_{rec}$ , i.e. the value  $< W - W_0 >_p$ , covariant graphs are used, as explained in section 3. In covariant gauge there are only 4 essentially different two-loop "recoil" graphs, whereas in the Coulomb gauge there are 24 ones (regarding magnetic and Coulomb quanta as depicted by different lines); here term "essentially different" mean that these graphs cannot be obtained from each other by transposition of two particles and/or time reversal.

The hard contribution to  $\Delta\nu_{rec}$  was evaluated in two ways. The first way is to use the Coulomb gauge and to evaluate contributions of the 24 graphs separately. The second one is to calculate  $< W - W_0 >_p$  as a whole using the Feynman gauge; in this case the  $W'$  is sum of contributions of only 4 graphs. Besides that, these contributions have simpler form than those of the Coulomb gauge (strictly speaking, the "simpler form" refers to simpler form of the program calculating these contributions).

The result for the "hard" contribution is  $-0.424(6)m\alpha^6$  for the "separate" and  $-0.426(6)m\alpha^6$  for the "united" calculation. Adding the "soft" contribution one obtains

$$\Delta\nu_{rec} = m\alpha^6 \left( -\frac{1}{6}\ln\alpha + 0.382(6) \right) \quad (68)$$

and

$$\Delta\nu_{rec} = m\alpha^6 \left( -\frac{1}{6}\ln\alpha + 0.380(6) \right) \quad (69)$$

for the "separate" and the "united" calculation methods respectively. These results are in perfect agreement with those found in [1, 2, 3].

The error estimation quoted in (68,69) is arbitrary in some extent, being subject of handwork; the reason of this is that method used in the program for calculation of  $W'$  is numerically unstable, and the error quoted arises from this instability (and not from the error of numerical integration). The direct way of the error estimation is to compare the results of "separate" and "united" calculations; difference between them is indeed within the estimation quoted above, as well as their differences from the results of works [1, 2, 3].

Results for contributions of separate graphs are quoted in tables 1a,1b,2 together with the revised results of paper [3], which agree with the results of the present paper. Errors are not given in the tables because they are not well defined; in fact, error for any contribution is  $\sim 1\%$ .

Notations for the graphs are following. If one makes no difference between magnetic and Coulomb lines, there exist 4 essentially different two-loop recoil graphs, denoted as 1,2,3,4 at fig. 5. To denote magnetic and Coulomb lines, indices "M" and "C" are written in the sequence the lines are attached to the lower fermion line from left to right. Values quoted in the tables refer to "essentially different" graphs only, including in each quoted value contributions of all graphs of the similar form.

Having been averaged over the full spin direction,  $W'$  depends on three variables; convenient choice for these variables are absolute values of the loop momenta, referred as  $p_1, p_2, q$ . For the "united" method of the calculation it is convenient to symmetrize  $W'$  over permutations of  $p_1, p_2, q$ , and to take into account only contributions of "essentially different" graphs, taking some of them multiplied by 2 or 4 because of transposition of the two particles and/or time reversal.

Let us describe briefly the checking of the calculation of the "hard" contributions. The first method of the checking is to compare  $\langle W - W_0 \rangle_p$  found in the two ways, the "separate" and the "united" (and intermediate comparison of  $W'(p_1, p_2, q)$  obtained in the two ways, at several points in  $p_1, p_2, q$ ).

Second, contributions to  $W'_0$  were found by a relatively short analytic calculation (except contributions associated with sum of graphs 1-CMM and 2-CMM), whereas contributions to  $W'$  are results of the numerical computing. An error in  $W'$  or  $W'_0$  leads usually to divergency of the numerical integration; clearly search for such errors is more convenient for the "separate" calculation.

Third, the results of the "separate" calculation are compared with the results of [3] in which  $\Delta\nu_{rec}$  was calculated using Bethe-Salpeter formalism and the Coulomb gauge. The

correction to the energy can be written as an expansion in irreducible covariant graphs (except the tree graph with Coulomb photon). Numerical value of any term of this expansion is clearly independent of formalism used, e.g. the formalism of the present paper or that of [3]. So most of contributions found in the present paper should be equal to corresponding contributions found in [3]. The other terms can be divided into groups, and total contribution of each group should also be equal to corresponding contribution of [3]. Besides that, to order under the consideration the contribution of graph 1-MCM should be equal separately to corresponding contribution of [3].

Results for "hard" contributions of separate graphs are given in tables 1a,1b. Part of them (namely, those quoted in table 1b)) are not uniquely defined as they depend on the choice of  $W_0$ . On the other hand, contribution of every term of the expansion in irreducible graphs, mentioned above, does not depend on formalism used, e.g. on the choice of  $W_0$ . Some terms of this expansion are trivial, i.e. they contain only "hard" parts, and every of them consist of contribution of just one covariant graph; corresponding results are quoted in table 1a. The other ("non-trivial") terms of this expansion involve also "soft" contributions, and their "hard" parts may consist of contributions of several covariant graphs; their values are given in table 2. Tables 1a,2 also contain corresponding results of [3]. All the results of [3] are in good agreement with those of the present work.

Contributions of graphs 3-MMC and 4-MMC were computed together, because these graphs contain noncovariant graphs shown in fig. 3(a) and 3(b), contributions of which to  $V_{30}$  cancel each other, which simplified the calculation. Contributions of graphs 1-CMM and 2-CMM were also evaluated together due to the similar reason.

In table 2 the value  $E_{1CCC}^H$ , for instance, denotes the "hard" part of the contribution of graph 1-CCC, i.e. one of values quoted in table 1b. The value  $E_M$  stands for

$$E_{3CCM}^H + E_{2CMC}^H + E_{1MCC}^H + E_{1CMC}^H + E_{LM}^S + E_M^S + E_P^S ,$$

and the  $E'_M$  denotes

$$(ct0+tc0)+ctc0+(ctx+tcx)+(cct0+tcc0)+(ctcy+ccty)+tccz+\Delta E_{MP}^{hfs}(\delta K_0 \cdot T + T \cdot \delta K_0) + \Delta E_d^{hfs}$$

(using notations of paper [3]).

Table 1a.  
 ”Trivial” (i.e. containing only the ”hard” part) contributions to  $\Delta\nu_{rec}$   
 (in units  $m\alpha^6$ ); the results of [3] are also quoted.

Graph	contribution to $\Delta\nu_{rec}$	(notations of [3])	contribution to $\Delta\nu_{rec}$	(obtained in [3])
4-CCC	-0.0039	ccc <sub>x</sub>	-0.0039	
4-CMC	0.0042	ctc <sub>x</sub>	0.0043	
4-MCC	-0.0486	cct <sub>x</sub> + tcc <sub>x</sub>	-0.0489	
4-MCM	-0.0230	tct <sub>x</sub>	-0.0230	
1-MMM	0.0694	ttt <sub>0</sub>	0.0694	
2-MMM	-0.0011	ttt <sub>y</sub>	-0.0011	
3-MMM	-0.0012	ttt <sub>z</sub>	-0.0011	
4-MMM	0.0042	ttt <sub>x</sub>	0.0041	
3-CCC	0.0064	ccc <sub>z</sub>	0.0063	
3-MCC	0.0268	ctcz+cct <sub>z</sub>	0.0283	
3-CMM	0.0530	ttcz+tct <sub>z</sub>	0.0534	
2-CCC	-0.0184	cccy+cc <sub>x</sub>	-0.0186	
2-MCC	-0.0661	tcc <sub>y</sub>	-0.0681	
2-MCM	0.0552	tct <sub>y</sub> +ttc <sub>y</sub>	0.0558	

Table 1b.

”Hard” parts of ”non-trivial” contributions to  $\Delta\nu_{rec}$  (in units  $m\alpha^6$ ).

Graph	
1-CCC	-0.0094
1-MCM	-0.0745
1-CMM + 2-CMM	-0.5112
4-MMC + 3-MMC	-0.0209
3-CCM	-0.0104
2-CMC	-0.0245
1-MCC	0.1795
1-CMC	-0.0092

Table 2.

”Non-trivial” contributions to  $\Delta\nu_{rec}$  (in units  $m\alpha^6$ ),  
compared with the results of [3].

value quoted	value quoted (notations of [3])	coefficient at $\ln \alpha$	constant	constant (obtained in [3])
$E_{4MMC+3MMC}^H$	(cttx+ttcx)+cttz	0	-0.0209	-0.0209
$E_{1CCC}^H + E_C^S$	cc0+ccc0	-1/48	-0.0146	-0.0148
$E_{1MCM}^H + E_{MCM}^S$	tct0	-5/48	0.3176	-0.3138
$E_{1CMM+2CMM}^H + E_{MM}^S$	ctt0+ttc0+ctty + + tt0+ttx	-3/8	-0.751	-0.749
$E_M$	$E'_M$	1/3	0.427	0.423

## 6. SUMMARY AND DISCUSSION.

In the present paper the ”recoil” contribution of order  $m\alpha^6$  to the hyperfine splitting of positronium ground state has been found. The calculation was performed using noncovariant formulation of QED perturbation theory. The result equals

$$\Delta\nu_{rec} = m\alpha^6 \left( -\frac{1}{6} \ln \alpha + 0.381(6) \right)$$

and is in perfect agreement with the results of [1, 2, 3]. Let us note that methods used in [1, 2] and in the present work are similar, whereas that of [3] is completely different. So  $\Delta\nu_{rec}$  may be considered to be firmly established. However agreement with the experimental data is poor. Combining nonlogarithmic part of the recoil contribution with all other contributions of the relevant order (they also may be considered to be reliable), one obtains the complete nonlogarithmic contribution  $\sim m\alpha^6$  to  $\Delta\nu$ :

$$m\alpha^6 ( -0.3928 ) = - 7.33 \text{ MHz} \quad (70)$$

((70) was obtained using for  $\Delta\nu_{rec}$  the result of [2], as the most accurate). The complete theoretical result up to the order  $m\alpha^6$  is sum of (70) and (3), and equals

$$\Delta\nu_{th} = 203\,392.96 \text{ MHz} .$$

It differs from the experimental result (2) by 5 standard deviations.

The leading term of the next order in  $\alpha$ , i.e. of order  $\sim m\alpha^7 \ln^2 \alpha$ , was found in [28] and equals

$$- \frac{7}{8\pi} m\alpha^7 \ln^2 \alpha = -0.92 \text{ MHz} .$$

Taking this correction into account reduces the difference between experimental and theoretical results to 4 standard deviations. One cannot reject possibility that this difference may be explained by contributions  $\sim m\alpha^7 \ln \alpha$ ,  $\sim m\alpha^7$  ).

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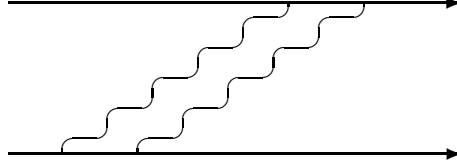


Fig. 1

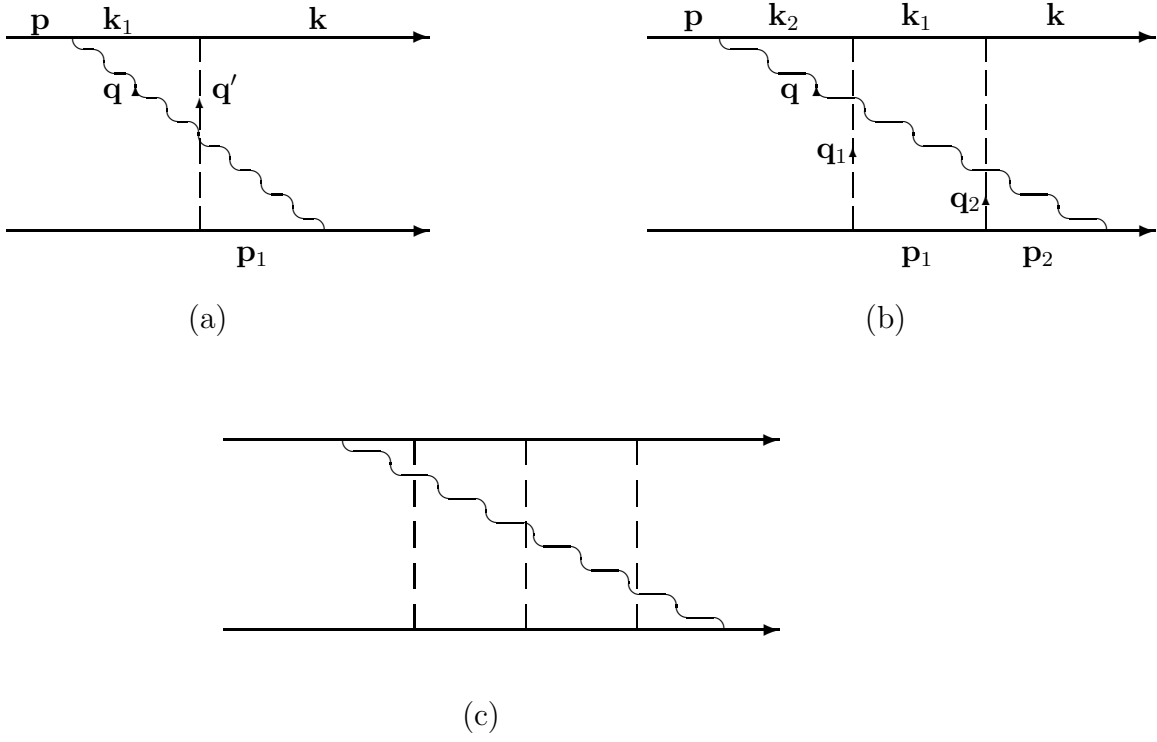


Fig.2

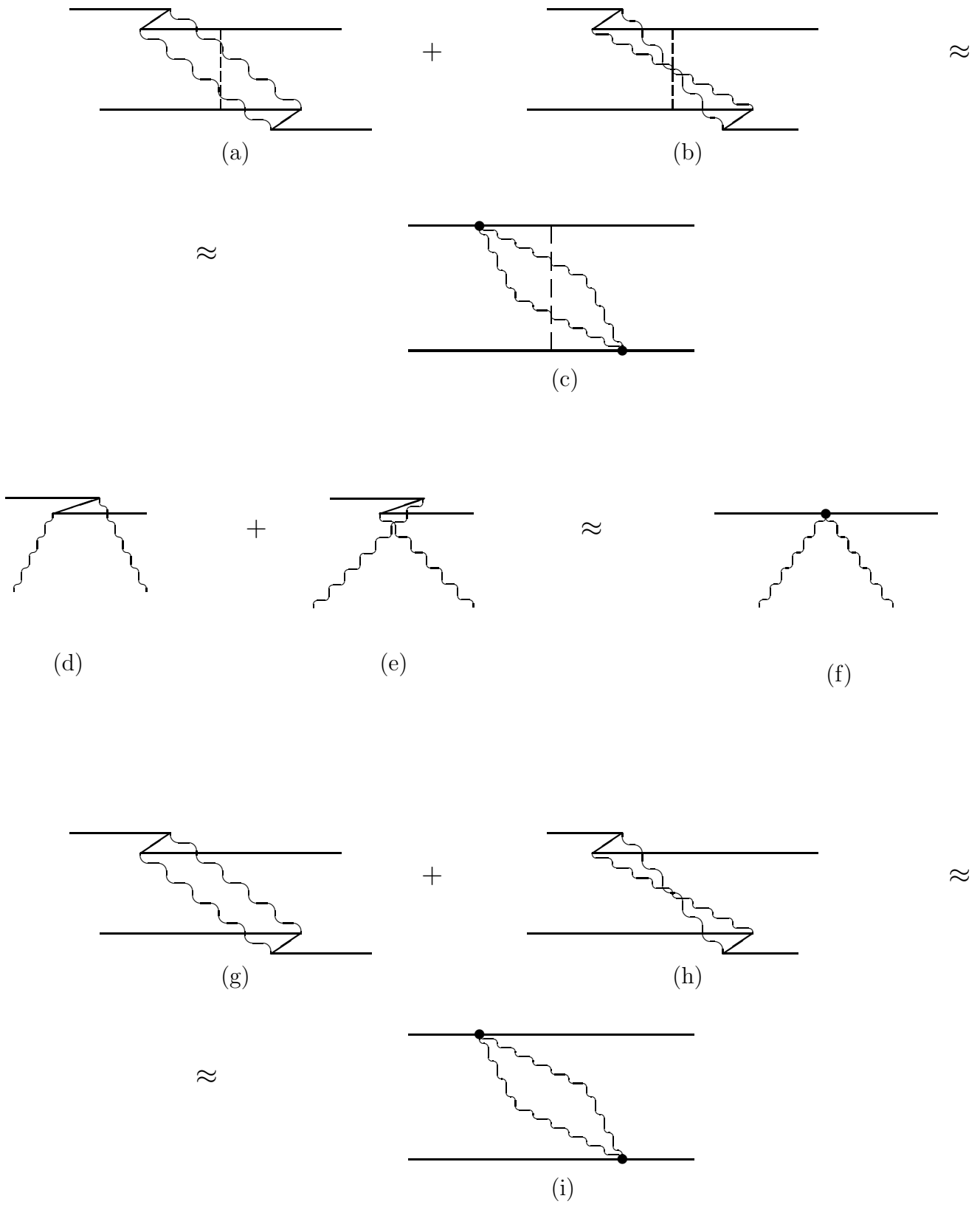
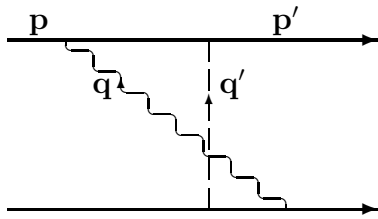
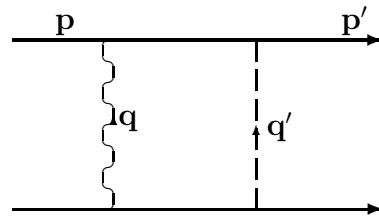


Fig. 3

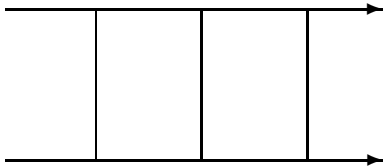


(A)

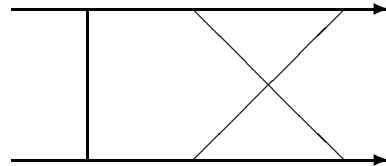


(B)

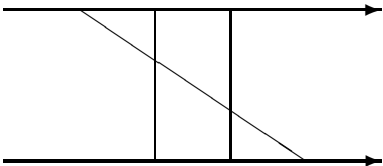
Fig. 4



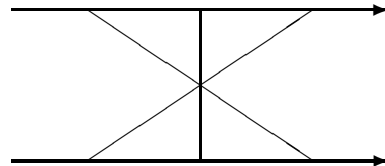
(1)



(2)



(3)



(4)

Fig. 5